Fluctuation-dissipation relations and universal behavior for relaxation processes in systems with static disorder and in the theory of mortality

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A unified description for the parallel relaxation in systems with static disorder and for the competitive risk mortality theory in population biology is suggested by combining the physical and biological approaches presented in the literature. A multichannel parallel decay process is investigated by assuming that each channel is characterized by a state vector $\mathbf{x}$ and by a probability of decaying $p(\mathbf{x};t)$. A general fluctuation-dissipation relation is derived which relates the effective decay rate of the process to the fluctuations of the density of channels characterized by different state vectors. A limit of the thermodynamic type in $\mathbf{x}$ space is introduced for which both the volume available and the average number of channels tend to infinity, but the average volume density of channels remains constant. By using scaling arguments combined with a stochastic renormalization group approach, two types of universal laws are identified in the thermodynamic limit for the relaxation (survival) function corresponding to nonintermittent and intermittent fluctuations of the density of channels, respectively. For nonintermittent fluctuations the general relaxation equation of Huber is recovered, which includes the stretched exponential equation as a particular case, whereas for intermittent fluctuations a more complicated universal relaxation equation is obtained which includes Huber's equation, the stretched exponential, and the inverse power law relaxation equations as particular cases. [S1063-651X(96)07805-1]

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Although models of relaxation processes in disordered systems [1–3] share some common features with the stochastic theory of mortality [4–6], there has been almost no interaction between these two branches of physics and biology. The purpose of this paper is to suggest a unified approach for the theory of parallel relaxation in systems with static disorder and the model of competitive risks in mortality theory by combining the different methods developed for the study of these two problems in the physical and biological literature.

We study a multichannel relaxation process which can take place by following different pathways (reaction channels) characterized by different values of an M-dimensional state vector $\mathbf{x}=(x_1,\ldots,x_M)$. For a physical relaxation process the state vector $\mathbf{x}$ may be the displacement vector between two interacting molecules or an individual relaxation rate [3], whereas in the case of the mortality process $\mathbf{x}$ is the vector of the relevant variables characterizing the state of an individual [7]. We denote by $p(\mathbf{x};t)$ the instantaneous probability of relaxation (death) at time t attached to an individual channel characterized by the state vector $\mathbf{x}$. The relaxation or the death occur if at least one of the individual channels lead to these processes. Denoting by $\mathscr{E}$ the total instantaneous probability of relaxation at time t, we have

$$\mathscr{E}[\zeta(\mathbf{x});t]\cong 1-\prod_u [1-p(\mathbf{x}_u;t)]^{\zeta(\mathbf{x}_u)\Delta\mathbf{x}_u}$$
$$=1-\exp\left\{\int \zeta(\mathbf{x})d\mathbf{x}\,\ln[1-p(\mathbf{x};t)]\right\}, \quad (1)$$

where $\zeta(\mathbf{x})d\mathbf{x}$ is the number of channels with a state vector between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$.

As the system is disordered, the state density of channels $\zeta(\mathbf{x})$ is a random function whose stochastic properties can be characterized by the characteristic functional

$$G[K(\mathbf{x})]=\left\langle \exp\left(i\int K(\mathbf{x})\zeta(\mathbf{x})d\mathbf{x}\right)\right\rangle, \quad (2)$$

where $K(\mathbf{x})$ is a test function conjugate to the density of states $\zeta(\mathbf{x})$. Herewith we restrict ourselves to systems with static disorder, for which the characteristic functional $G[K(\mathbf{x})]$ is time independent. This assumption of time independence is also consistent with the theory of competitive risks in population dynamics, where the contributions of the different factors to the mortality process are assumed to be time independent [4–6].

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The observable function both in solid-state physics and in population biology is the overall probability of relaxation (death),

$$\langle \mathscr{E}(t)\rangle = \langle \mathscr{E}[\zeta(\mathbf{x});t]\rangle, \tag{3}$$

or the complementary probability

$$l(t) = 1 - \langle \mathscr{E}(t)\rangle, \tag{4}$$

which bear the names of relaxation function and survival function in physics and in biology, respectively. An alternative characterization of the process can be given in terms of the effective relaxation rate (the mortality force)

$$\mu(t) = -\partial_t \ln l(t). \tag{5}$$

In Eq. (3) the average is taken over all possible values of the density of states $\zeta(x)$. It is easy to check that this average can be expressed in terms of the characteristic functional $G[K(\mathbf{x})]$. By combining Eqs. (2)–(4), we obtain

$$l(t) = G[K(\mathbf{x}) = ib(\mathbf{x};t)], \tag{6}$$

where

$$b(\mathbf{x};t) = -\ln[1 - p(\mathbf{x};t)] \tag{7}$$

is the bit number [8] of the individual probability of survival $1-p(x;t)$ attached to the channel characterized by the state vector $\mathbf{x}$.

If the cumulants of the density of states $\zeta(x)$, $\langle\langle l(x_1)\ldots l(x_m)\rangle\rangle$, $m=1,2,...$, which describe the fluctuations of the number of channels with different state vectors, exist and are finite, we can express the characteristic functional $G[K(\mathbf{x})]$ as [9]

$$G[K(\mathbf{x})] = \exp\left\{\sum_{m=1}^{\infty} \frac{i^m}{m!} \int \cdots \int \langle\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_m)\rangle\rangle K(\mathbf{x}_1)\cdots K(\mathbf{x}_m) d\mathbf{x}_1 \cdots d\mathbf{x}_m\right\}, \tag{8}$$

from which we obtain the following expressions for the relaxation function $l(t)$ and for the effective relaxation rate $\mu(t)$:

$$l(t) = \exp\left\{\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int \cdots \int \langle\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_m)\rangle\rangle b(\mathbf{x}_1;t)\cdots b(\mathbf{x}_m;t) d\mathbf{x}_1 \cdots d\mathbf{x}_m\right\}, \tag{9}$$

$$\mu(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} \int \cdots \int \langle\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_m)\rangle\rangle \left[\prod_{u=1}^{m} b(\mathbf{x}_u;t)\right] \partial_t \ln\left[\prod_{u=1}^{m} b(\mathbf{x}_u;t)\right] d\mathbf{x}_1 \cdots d\mathbf{x}_m . \tag{10}$$

Equations (9) and (10) are general fluctuation-dissipation relations which relate the average time-dependent behavior of the system expressed by the functions $l(t)$ or $\mu(t)$ to the fluctuations of the numbers of channels characterized by different state vectors expressed by the cumulants $\langle\langle \zeta(\mathbf{x}_1)...\zeta(\mathbf{x}_m)\rangle\rangle$. Although valid both in the physical and biological contexts considered in this paper, Eqs. (9) and (10) are not very useful because they depend on a number of unknown functions. That is why in the following we investigate the possibility of an occurrence of certain types of universal limit behaviors in the limit corresponding to a very large number of channels.

We start out by considering that the possible values of the state vector $\mathbf{x}$ belong to a certain domain Σ of the $\mathbf{x}$ space. The corresponding volume V_Σ and the total average number of channels $\langle \mathscr{N}\rangle$ are given by

$$V_\Sigma = \int_\Sigma d\mathbf{r}, \tag{11}$$

$$\langle \mathscr{N}\rangle = \int_\Sigma \langle \zeta(\mathbf{x})\rangle d\mathbf{x} = \int_\Sigma \langle\langle \zeta(\mathbf{x})\rangle\rangle d\mathbf{x}, \tag{12}$$

where we have used the property that the first cumulant $\langle\langle \zeta(\mathbf{x})\rangle\rangle$ of the density of states $\zeta(\mathbf{x})$ is equal to the corresponding average value $\langle \zeta(\mathbf{x})\rangle$. We consider a limit of the thermodynamic type for which both the average number of channels $\langle \mathscr{N}\rangle$ and the volume V_Σ of the available state space tend to infinity but the average volume density of channels

$$\epsilon = \langle \mathscr{N}\rangle / V_\Sigma \tag{13}$$

remains constant:

$$V_\Sigma, \langle \mathscr{N}\rangle \to \infty \quad \text{with} \quad \epsilon = \langle \mathscr{N}\rangle / V_\Sigma = \text{const.} \tag{14}$$

Such a limit has been recently introduced in a biological context for the study of space-dependent epidemics [10]. For investigating the asymptotic behavior which emerges in the limit (14), we should have some knowledge concerning the nature of the fluctuations of the number of channels. We introduce the relative fluctuations of different orders $m=2,3,...$,

$$c_m(\mathbf{x}_1,\ldots,\mathbf{x}_m) = \frac{\langle\langle \zeta(\mathbf{x}_1)\ldots\zeta(\mathbf{x}_m)\rangle\rangle}{\prod_{u=1}^{m}\langle\langle \zeta(\mathbf{x}_u)\rangle\rangle}, \quad m=2,3,\ldots . \tag{15}$$

If the functions $c_m(\mathbf{x}_1,...,\mathbf{x}_m)$ decrease to zero in the thermodynamic limit (14),

$$c_m(\mathbf{x}_1,\ldots,\mathbf{x}_m) \to 0 \quad \text{as} \quad V_\Sigma, \langle \mathscr{N}\rangle \to \infty \quad \text{with} \quad \epsilon = \text{const}, \tag{16}$$

then the fluctuations of the number of channels are nonintermittent; otherwise, if in the thermodynamic limit the functions $c_m(\mathbf{x}_1,...,\mathbf{x}_m)$ do not decrease to zero but tend toward constant values different from zero or diverge to infinity, then the fluctuations of the number of channels are intermittent.

For investigating the asymptotic behavior in the thermodynamic limit (14), we need to know the dependence of the individual probability of relaxation $p(\mathbf{x};t)$ attached to a given channel of the state vector $\mathbf{x}$ and of the time t. We denote by $W(\mathbf{x})$ the individual relaxation (death) rate of an individual channel characterized by the state vector $\mathbf{x}$ and by $\lambda(\mathbf{x})$ the probability that the channel is open. If an individual channel were always open $[\lambda(\mathbf{x})=1]$ or closed $[\lambda(\mathbf{x})=0]$, we would have

$$1-p(\mathbf{x};t)=\exp[-tW(\mathbf{x})] \quad \text{for } \lambda(\mathbf{x})=1, \tag{17}$$

$$1-p(\mathbf{x};t)=1 \quad \text{for } \lambda(\mathbf{x})=0. \tag{18}$$

For a probability $\lambda(\mathbf{x})$ between zero and unity, the probability $1-p(\mathbf{x};t)$ is an average of the values corresponding to the two limit cases given by Eqs. (17) and (18):

$$1-p(\mathbf{x};t)=\lambda(\mathbf{x})\exp[-tW(\mathbf{x})]+1-\lambda(\mathbf{x}). \tag{19}$$

Concerning the probability $\lambda(\mathbf{x})$ that the channel characterized by the state vector $\mathbf{x}$ is open, we assume that it is the ratio between a characteristic volume $V^*(\mathbf{x})$ of a neighborhood of the state $\mathbf{x}$, and the total volume V_Σ available in the $\mathbf{x}$ space,

$$\lambda(\mathbf{x})=V^*(\mathbf{x})/V_\Sigma . \tag{20}$$

Equation (20) expresses the locality of the behavior of channels; a similar relationship has been suggested in the context of the theory of epidemics [10]. An important feature of the instantaneous decay law (19) is the assumption of exponential relaxation for the case when the channel is open; in its present form this assumption was introduced by Huber ten years ago [11]; he showed that it may be viewed as being a result of a local Markovian behavior of the different individual channels.

By combining Eqs. (9)–(19) we arrive at the following expression for the relaxation (survival) function $l(t)$:

$$l(t)=\exp\left\{\sum_{m=1}^{\infty}\frac{\epsilon^m}{m!}\int_\Sigma\cdots\int_\Sigma c_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)\xi(\mathbf{x}_1)\cdots\xi(\mathbf{x}_m)\times\prod_{u=1}^{m}\left\{V_\Sigma\ln\left[1-\frac{V^*(\mathbf{x}_u)}{V_\Sigma}\times\{1-\exp[-tW(\mathbf{x}_u)]\}\right]\right\}d\mathbf{x}_1\cdots d\mathbf{x}_m\right\}, \tag{21}$$

where

$$\xi(\mathbf{x})d\mathbf{x}=\langle\langle\zeta(\mathbf{x})\rangle\rangle d\mathbf{x}\Bigg/\int_\Sigma\langle\langle\zeta(\mathbf{x})\rangle\rangle d\mathbf{x} \quad \text{with} \int_\Sigma\xi(\mathbf{x})d\mathbf{x}=1, \tag{22}$$

is the average probability that the state vector of an individual channel is between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ and

$$c_1(\mathbf{x})=1 \quad \text{independent of } \mathbf{x}. \tag{23}$$

By assuming that the fluctuations of the number of channels are nonintermittent, and that the conditions of nonintermittency (16) hold in the thermodynamic limit (14), Eq. (21) leads to the universal behavior

$$l(t)=\exp\left\{-\int\rho(W)[1-\exp(-Wt)]dW\right\}, \tag{24}$$

where

$$\rho(W)=\epsilon\int\delta[W-W(\mathbf{x})]V^*(\mathbf{x})\xi(\mathbf{x})d\mathbf{x} \tag{25}$$

is the average density of active channels distributed according to their relaxation rates. Relationship (24) was derived in the physical literature by Huber [11] on the basis of a model for the decay of luminiscence. An alternative derivation of Eq. (24) based on the use of the theory of random point processes has recently been suggested by two of the present authors [12]. Our derivation of Huber's law (24) is less restrictive than the other proofs presented in the literature, because it is not based on a particular model but is rather a universal law which emerges in the thermodynamic limit (14) in the case of nonintermittent fluctuations of the number of channels.

The study of the universal law which emerges in the case of intermittent fluctuations is more complicated. In this case a renormalization group technique should be used. In the following we apply a probabilistic version [13] of the Shlesinger-Hughes stochastic renormalization procedure [14] which was recently applied to a study of space-dependent epidemics with high migration [10]. The method consists in starting from an initial characteristic functional $G[K(\mathbf{x})]$ of the density of states $\zeta(\mathbf{x})$ for which the fluctuations are nonintermittent and in constructing, by means of a succession of decimation processes, a renormalized characteristic functional $\tilde{G}[K(\mathbf{x})]$ for which the fluctuations of the density of states are intermittent. The main steps of such an approach are presented in another context in Ref. [13], and a simplified derivation is also presented in Ref. [10]. The corresponding expression for the renormalized characteristic functional $\tilde{G}[K(\mathbf{x})]$ is given by (see Appendix A)

$$\tilde{G}[K(\mathbf{x})]=H\int_0^1 z^{H-1}G[-i\ln[1-z\times\{1-\exp(iK(\mathbf{x}))\}]]dz, \quad H>0, \tag{26}$$

where $H>0$ is a positive fractal exponent which describes the intermittent nature of the fluctuations of the number of channels. By expressing the renormalized cumulants $\langle\langle\tilde{\zeta}(\mathbf{x}_1)\cdots\tilde{\zeta}(\mathbf{x}_m)\rangle\rangle$ and the corresponding nonrenormalized cumulants $\langle\langle\zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_m)\rangle\rangle$ as functional derivatives of the characteristic functionals $\tilde{G}[K(\mathbf{x})]$ and $G[K(\mathbf{x})]$, respectively,

$$\langle\langle\tilde{\zeta}(\mathbf{x}_1)\cdots\tilde{\zeta}(\mathbf{x}_m)\rangle\rangle=(-i)^m\delta^m\ln\tilde{G}[K(\mathbf{x})=0]/[\delta K(\mathbf{x}_1)\cdots\delta K(\mathbf{x}_m)], \tag{27}$$

$$\langle\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_m)\rangle\rangle$$
$$=(-i)^m\delta^m \ln G[K(\mathbf{x})=0]/[\delta K(\mathbf{x}_1)\cdots\delta K(\mathbf{x}_m)], \quad (28)$$

it is easy to check that, if the nonrenormalized fluctuations are nonintermittent and obey the nonintermittency conditions (16), then the renormalized fluctuations are intermittent.

For computing the universal law which emerges in the thermodynamic limit (14) for intermittent fluctuations, in Eq. (26) we expand the nonrenormalized characteristic functional $G[K(\mathbf{x})]$ in the cumulant expansion (8) and express the nonrenormalized cumulants $\langle\langle\zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_m)\rangle\rangle$ in terms of the nonrenormalized relative fluctuations $c_m(\mathbf{x}_1,...,\mathbf{x}_m)$ given by Eqs. (15) and in terms of the average renormalized density of channels,

$$\epsilon=\langle\tilde{\mathcal{N}}\rangle/V_\Sigma=\frac{H}{H+1}\frac{\langle\mathcal{N}\rangle}{V_\Sigma}. \quad (29)$$

Here we have used the relationship between the renormalized and nonrenormalized average number of channels,

$$\langle\tilde{\mathcal{N}}\rangle=\langle\mathcal{N}\rangle H/(H+1), \quad (30)$$

which can easily be derived by the functional differentiation of Eq. (26) followed by the application of Eqs. (27) and (28) for $m=1$, and by the integration over the state vector $\mathbf{x}$. After lengthy algebraic manipulations Eq. (6) for the average relaxation function $l(t)$ can be written as

$$l(t)=H\int_0^1 z^{H-1}dz\ \exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}[\epsilon(1+1/H)]^m\int_\Sigma\cdots\int_\Sigma c_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)\xi(\mathbf{x}_1)\cdots\xi(\mathbf{x}_m)\right.$$
$$\left.\times\prod_{u=1}^{m}\left\{V_\Sigma\ \ln\left[1-z\frac{V^*(\mathbf{x}_u)}{V_\Sigma}\{1-\exp[-tW(\mathbf{x}_u)]\}\right]\right\}d\mathbf{x}_1\cdots d\mathbf{x}_m\right\}. \quad (31)$$

By passing in Eq. (31) to the thermodynamic limit (14) and using the nonintermittency conditions (16) for the nonrenormalized fluctuations, we obtain the universal law

$$l(t)=j_H\left[\int\rho(W)[1-\exp(-Wt)]dW\right], \quad (32)$$

where the function $j_H(z)$ can be expressed in terms of the complete gamma function

$$\gamma(a,u)=\int_0^u t^{a-1}\exp(-t)dt,\quad a>0,\quad u\geqslant 0. \quad (33)$$

We have

$$j_H(z)=H[(1+1/H)z]^{-H}\gamma[H,(1+1/H)z]. \quad (34)$$

Now we consider certain particular cases of the universal laws (24) and (32) of the relaxation function $l(t)$ derived for nonintermittent and intermittent fluctuations, respectively. It is easy to check that the intermittent law (32) includes the Huber's equation (24) as a particular case corresponding to the limit $H\to\infty$. We can show that in this limit the function $j_H(z)$ tends toward an exponential:

$$\lim j_H(z)=\exp(-z),\quad H\to\infty, \quad (35)$$

and Eq. (32) reduces to Eq. (24). The physical interpretation of this result is simple. The reciprocal value of the fractal exponent,

$$\varphi=1/H, \quad (36)$$

is a measure of the intermittency of the fluctuations of the number of channels: as the fractal exponent H increases, the intermittent character of the fluctuations becomes less and less pronounced, and in the limit $H\to\infty$ ($\varphi\to 0$) it vanishes completely, resulting in the nonintermittent law (24).

A particular case of importance both for physics and biology is the stretched exponential survival statistics for which the relaxation function $l(t)$ is given by

$$l(t)=\exp[-(\Omega t)^\alpha],\quad 1>\alpha>0, \quad (37)$$

where Ω is a characteristic frequency and α is a fractal exponent between zero and unity. Equation (37) describes a broad class of relaxation phenomena in condensed matter physics [1–3], and, on the other hand, gives a representation of the survival function of cancer patients [7,15]. The effective relaxation rate $\mu(t)$ corresponding to the stretched exponential (37) is

$$\mu(t)=\alpha\Omega(\Omega t)^{\alpha-1}. \quad (38)$$

For establishing the conditions within which the stretched exponential (37) emerges as a particular case of the universal laws (24) or (32), we rewrite the fluctuation-dissipation relation (10) in the thermodynamic limit (14); the corresponding expressions for nonintermittent and intermittent fluctuations, respectively, are

$$\mu(t)=\int W\rho(W)\exp(-Wt)dW, \quad (39)$$

$$\mu(t)=\beta_H\left[\int dW\rho(W)[1-\exp(-Wt)]\right]\int W\rho(W)$$
$$\times\exp(-Wt)dW, \quad (40)$$

where the function $\beta_H(\mathbf{x})$ is given by

$$\beta_H(x)=(H+1)x^{-1}\{1+[(1+1/H)^H x^H]\times\exp[-x(1+1/H)]/\gamma[H+1,x(1+1/H)]\}^{-1}. \tag{41}$$

As expected, as $H\to\infty$ we have

$$\lim_{H\to\infty}\beta_H(x)=1, \tag{42}$$

and Eq. (40) reduces to Eq. (39) for nonintermittent fluctuations.

If the effective relaxation rate is known from experiments, then Eqs. (39) and (40) can be considered as functional equations for the density of states $\rho(W)$ expressed in terms of the relaxation rate W. Equation (39) for nonintermittent fluctuations can be easily resolved by means of an inverse Laplace transformation. In particular, in the case of stretched exponential relaxation, for which the effective relaxation rate is given by Eq. (38), Eq. (39) leads to a negative power law for the density of states $\rho(W)$:

$$\rho(W)=\alpha\Omega^{\alpha}W^{-(1+\alpha)}/\Gamma(1-\alpha)$$
$$\text{with}\quad \Gamma(x)=\gamma(x,\infty),\quad x>0. \tag{43}$$

The functional Eq. (40) for intermittent fluctuations cannot be solved in a simple way. For a comparison with the nonintermittent case we use an inverse approach and evaluate the relaxation function $l(t)$ for the case when the density of states $\rho(W)$ is given by the negative power law (43). By inserting Eq. (43) into Eqs. (33) and (40), we obtain

$$l(t)=H(\Omega t)^{-\alpha H}(1+1/H)^{-H}\gamma[H,(\Omega t)^{\alpha}(1+1/H)] \tag{44}$$

and

$$\mu(t)=\alpha\Omega(\Omega t)^{\alpha-1}\beta_H[(\Omega t)^{\alpha}]. \tag{45}$$

From Eqs. (44) and (45) for small times, we recover the stretched exponential behavior

$$l(t)\sim\exp[-(\Omega t)^{\alpha}],\quad t\ll\Omega^{-1}, \tag{46}$$

$$\mu(t)\sim\alpha\Omega(\Omega t)^{\alpha-1},\quad t\ll\Omega^{-1}, \tag{47}$$

whereas for large times we obtain a negative power law

$$l(t)\sim\Gamma(1+H)(\Omega t)^{-\alpha H}(1+1/H)^{-H},$$
$$t\gg\Omega^{-1},\quad H\ \text{finite}, \tag{48}$$

$$\mu(t)\sim\alpha H/t,\quad t\gg\Omega^{-1},\quad H\ \text{finite}. \tag{49}$$

As the fractal exponent H increases, the stretched exponential portion of the relaxation function $l(t)$ becomes longer and longer and the power law tail becomes shorter and shorter; eventually in the limit $H\to\infty$ the whole relaxation function $l(t)$ can be represented by a stretched exponential.

It has been suggested that it would be interesting to investigate the connection between the overall relaxation (death) process and the fluctuations of the density of channels for nonintermittent and intermittent systems, respectively. Although, in general, according to the fluctuation-dissipation relations (9) and (10) the average behavior of the overall process is characterized by the relative fluctuations of all orders, the limit behavior for nonintermittent and intermittent systems, described by Eqs. (24) and (32) is independent of the relative fluctuations. It therefore seems that the fluctuations do not play any role in the thermodynamic limit; however, this is not the case. We should make a distinction between the total number of relaxation channels and the channels which are active; that is, those channels which are involved in the relaxation process. The average density of active channels involved in the relaxation process is given by Eq. (25) both for nonintermittent and intermittent fluctuations; their distribution is random and different in the two cases.

We can describe the properties of the random distribution of the active channels in terms of a set of grand canonical probability densities

$$Q_0,\quad Q_N(W_1,\ldots,W_N)dW_1\cdots dW_N, \tag{50}$$

with the normalization condition

$$Q_0+\sum_{N=1}^{\infty}\frac{1}{N!}\int\cdots\int Q_N(W_1,\ldots,W_N)dW_1\cdots dW_N. \tag{51}$$

$Q_N(W_1,\ldots,W_N)dW_1\ldots dW_N$ is the probability that there are N active channels and that their individual relaxation rates are between W_1 and W_1+dW_1,..., and W_N and W_N+dW_N, respectively. The total relaxation rate of the process is given by the sum of the individual rates corresponding to the different channels which are active:

$$W=W_1+\cdots+W_N, \tag{52}$$

and then the average survival function can be expressed as a grand canonical average of an exponential survival function $\exp(-Wt)$, where W is given by Eq. (52):

$$l(t)=Q_0+\sum_{N=1}^{\infty}\frac{1}{N!}\int\cdots\int\exp\left(-t\sum_{u=1}^{N}W_u\right)\times Q_N(W_1,\ldots,W_N)dW_1\cdots dW_N. \tag{53}$$

If in Eq. (53) the average survival function $l(t)$ is given by one of the two limit laws, (24) or (32), then this equation can be viewed as a functional equation for the grand canonical probability densities of the active channels. In Appendix B we show that for nonintermittent fluctuations the solution of this functional equation is

$$Q_0=\exp\left(-\int\rho(W)dW\right), \tag{54}$$

$$Q_N(W_1,\ldots,W_N)=\rho(W_1)\cdots\rho(W_N)\exp\left(-\int\rho(W)dW\right), \tag{55}$$

that is, the distribution of active channels is given by a Poissonian random point process. In particular, considering the number of active channels with rates between a minimum value $W_{\min}$ and a maximum value $W_{\max}$, the corresponding probability distribution is a Poissonian:

$$P(N)=\langle N\rangle^{N}(N!)^{-1}\exp(-\langle N\rangle), \tag{56}$$

where the average number of channels is given by

$$\langle N\rangle=\int_{W_{\min}}^{W_{\max}}\rho(W)dW. \tag{57}$$

Now the cause for the apparent absence of the fluctuations in the nonintermittent limit law is clear. For a Poissonian distribution all cumulants of a random variable are equal to the average value, and this is the reason why the limit equation (24) depends only on the average density of channels.

A similar analysis can be performed for intermittent fluctuations, resulting in the following expressions for the random distribution of the active channels (see Appendix B):

$$Q_0=H\int_0^1 z^{H-1}dz\ \exp\left(-z(1+1/H)\int\rho(W)dW\right)=j_H\left(\int\rho(W)dW\right), \tag{58}$$

$$\begin{aligned}Q_N(W_1,\ldots,W_N)&=\rho(W_1)\cdots\rho(W_N)H(1+1/H)^N\\&\quad\times\int_0^1 z^{H+N-1}dz\ \exp\left(-z(1+1/H)\times\int\rho(W)dW\right)\\&=H(1+1/H)^{-H}\gamma\left(H+N,(1+1/H)\times\int\rho(W)dW\right)\\&\quad\times\left(\int\rho(W)dW\right)^{-N-H}\rho(W_1)\cdots\rho(W_N).\end{aligned} \tag{59}$$

In this case, due to the intermittent behavior, the distribution of active channels is no longer Poissonian; however, since it can be represented as a superposition of Poissonians the fluctuation dynamics is entirely characterized by the average density of channels $\rho(W)$ and by the fractal exponent H. This fact explains why the intermittent limit law (32) seems to be apparently independent of fluctuations.

We conclude this paper by outlining some general features of our approach, as well as some of its limitations and possibilities of generalization. The occurrence of the asymptotic universal laws (24) and (32) for the relaxation function $l(t)$ is due to two different properties of the process. The first property is related to the nonintermittent or intermittent character of the fluctuations of the number of channels, whereas the second property consists in the local behavior of an individual channel in $\mathbf{x}$ space, expressed by the finiteness of the volume $V^*(\mathbf{x})$ of the neighborhood of the state $\mathbf{x}$ which corresponds to the open state of the channel. Note that this local behavior is conserved in the thermodynamic limit (14) because in Eqs. (21) and (31) for nonintermittent and intermittent fluctuations, respectively, $V^*(\mathbf{x})$ is assumed to be constant when V_Σ and $\langle\mathcal{N}\rangle$ tend to infinity.

Limit (14) of the thermodynamic type expresses the fact that the process considered is complex and involves a very large number of pathways (channels) which are uniformly and randomly distributed in the state space. Although sufficient for the occurrence of the two types of universal laws (24) and (32) for nonintermittent and intermittent fluctuations, the assumptions made in this paper are incomplete, and cannot be used for specifying the density of active channels $\rho(W)$ with a rate between W and $W+dW$ and the fractal exponent H. This is both an advantage and a disadvantage of our approach: it is the main reason for which the results are valid both for biological and physical systems and, on the other hand, because of its incompleteness, our formalism is not a theory but rather a scenario which should be completed with specific assumptions describing the physical or biological processes considered. Such a development would lead to particular models for different physical or biological processes providing expressions for the density of states $\rho(W)$ and for the fractal exponent H, but of course the generality of the treatment would be lost.

Our approach is valid for systems with static disorder: for such systems once a fluctuation of the number of channels has occurred it is completely frozen and lasts forever. Such an assumption is justified if the characteristic time scale for the regression of fluctuations is much larger than the time scale of the process itself. If the two time scales have the same order of magnitude then the dynamic character of fluctuations should be taken into account; the fluctuations of the number of channels are continuously formed and destroyed, the characteristic functionals $G[K(\mathbf{x})]$ and $\tilde{G}[K(\mathbf{x})]$ also depend on time, and the average in Eq. (3) is dynamical, being taken over all possible random functions $\zeta(\mathbf{x};t)$. Further research concerning this dynamical problem is presented in Ref. [17].

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APPENDIX A

It has been suggested that the main steps of the derivation of the renormalized group equation (26) for intermittent fluctuations should be presented in an appendix. Here we give a simplified derivation of this equation. For further details the interested reader may consult Refs. [10] and [13].

The renormalization transformation generating a limit intermittent behavior consists in a succession of decimation processes of the number of channels characterized by two probabilities: the probability α that a decimation process takes place and the probability ν that for a given step a channel is decimated (left out). For characterizing the decimation process we use a discrete representation of the numbers of channels characterized by different state vectors, and denote by

$$N_u=\zeta(\mathbf{x}_u)\Delta\mathbf{x}_u,\quad u=1,2,\ldots \tag{A1}$$

the number of channels with a state vector between $\mathbf{x}_u$ and $\mathbf{x}_u+d\mathbf{x}_u$. We introduce the notation

$$P(N_1,N_2,\ldots) \quad \text{with} \sum_{N_1}\sum_{N_2}\cdots P(N_1,N_2,\ldots)=1 \tag{A2}$$

for the initial (nonrenormalized) probability that there are N_1 channels in the group u_1, N_2 channels in group u_2, etc.,

$$P_q(N_1,N_2,\ldots) \quad \text{with} \sum_{N_1}\sum_{N_2}\cdots P_q(N_1,N_2,\ldots)=1 \tag{A3}$$

and

$$\widetilde{P}(N_1,N_2,\ldots) \quad \text{with} \sum_{N_1}\sum_{N_2}\cdots \widetilde{P}(N_1,N_2,\ldots)=1, \tag{A4}$$

for the corresponding probabilities attached to the qth decimation step and to the renormalized process, respectively.

Since the probability α for the occurrence of a decimation step is constant, the probability χ_q that the renormalization process is made up of q decimation steps is given by a Pascal law,

$$\chi_q=\alpha^q(1-\alpha). \tag{A5}$$

Due to the independence of the different decimation processes, it is easier to focus on the description of the decimation of the channels of a given type. Since the decimation of a channel is a random process characterized by a constant probability, ν, we have

$$P_q(\ldots,N_v,\ldots)=\sum P_{q-1}(\ldots,N_v^{(q-1)},\ldots)\times\frac{N_v^{(q-1)}!}{N_v^{(q)}!(N_v^{(q-1)}-N_v^{(q)})!}\times\nu^{N_v^{(q-1)}-N_v^{(q)}}(1-\nu)^{N_v^{(q)}}, \tag{A6}$$

with the initial condition

$$P_0(\ldots,N_v,\ldots)=P(\ldots,N_v,\ldots). \tag{A7}$$

The renormalized distribution which eventually emerges after the completion of the succession of decimation processes can be evaluated from P by averaging over the number q of steps in terms of the probability χ_q:

$$\widetilde{P}(\ldots,N_v,\ldots)=\sum_{q=0}^{\infty}\alpha^q(1-\alpha)P_q(\ldots,N_v,\ldots). \tag{A8}$$

By introducing the generating functions

$$\vartheta(\ldots,y_v,\ldots)=\sum(1-y_v)^{N_v}P(\ldots,N_v,\ldots), \quad |1-y_v|\leqslant 1, \tag{A9}$$

$$\vartheta_q(\ldots,y_v,\ldots)=\sum(1-y_v)^{N_v}P_q(\ldots,N_v,\ldots), \quad |1-y_v|\leqslant 1, \tag{A10}$$

$$\widetilde{\vartheta}(\ldots,y_v,\ldots)=\sum(1-y_v)^{N_v}\widetilde{P}(\ldots,N_v,\ldots), \quad |1-y_v|\leqslant 1, \tag{A11}$$

Eqs. (A6)–(A11) lead to

$$\vartheta_q(\ldots,y_v,\ldots)=\vartheta_{q-1}[\ldots,(1-\nu)y_v,\ldots], \tag{A12}$$

with the initial condition

$$\vartheta_0(\ldots,y_v,\ldots)=\vartheta(\ldots,y_v,\ldots) \tag{A13}$$

and

$$\widetilde{\vartheta}(\ldots,y_v,\ldots)=\sum_{q=0}^{\infty}\alpha^q(1-\alpha)\vartheta_q(\ldots,y_v,\ldots). \tag{A14}$$

By applying Eqs. (A12)–(A14) recursively, we obtain

$$\widetilde{\vartheta}(\ldots,y_v,\ldots)=\sum_{q=0}^{\infty}\alpha^q(1-\alpha)\vartheta[\ldots,(1-\nu)^q y_v,\ldots]=(1-\alpha)\vartheta(\ldots,y_v,\ldots)+\alpha\widetilde{\vartheta}[\ldots,(1-\nu)y_v,\ldots]. \tag{A15}$$

Equation (A15) has a structure typical of a renormalization group equation [14]. It generates a negative power law scaling behavior for the generating function $\widetilde{\vartheta}$ in terms of y_{v} with a fractal exponent

$$H=\ln\alpha/\ln(1-\nu), \tag{A16}$$

modulated by logarithmic oscillations in $\ln y_v$ with period $-\ln(1-\nu)$. Since these logarithmic oscillations lead to a violation of the self-similarity of the process they should be discarded. To eliminate the logarithmic oscillations we consider the limit [16].

$$\alpha\nearrow 1, \nu\searrow 0 \quad \text{with} \quad H=\ln\alpha/\ln(1-\nu)=\text{const.} \tag{A17}$$

In this limit the logarithmic oscillations vanish, but the negative power law scaling behavior is still present. From Eqs. (A15)–(A17) we obtain the differential equation

$$y_v d\widetilde{\vartheta}/dy_v=H(\vartheta-\widetilde{\vartheta}). \tag{A18}$$

The solution of Eq. (17), which conserves the normalization conditions for the probabilities, has the form

$$\widetilde{\vartheta}(\ldots,y_v,\ldots)=H\int_0^1 z^{H-1}dz\,\vartheta(\ldots,zy_v,\ldots). \tag{A19}$$

The simplified derivation presented above describes the decimation process for only one type of channels. The generalization for many types of channels is straightforward. The detailed derivations are left to the reader. We mention only that Eqs. (A6), (A18), and (A19) are replaced by

$$P_q(N_1,N_2,\ldots)=\sum_{N_1'}\sum_{N_2'}\cdots\prod_m\left\{\frac{N_m'!}{N_m!(N_m'-N_m)!}\nu^{N_m'-N_m}\times(1-\nu)^{N_m}\right\}P_{q-1}(N_1',N_2',\ldots),\quad (A20)$$

$$\sum_m y_m\partial\tilde{\vartheta}/\partial y_m=H(\vartheta-\tilde{\vartheta}),\quad (A21)$$

and

$$\tilde{\vartheta}(y_1,y_2,\ldots)=H\int_0^1 z^{H-1}dz\,\vartheta(zy_1,zy_2,\ldots),\quad (A22)$$

where

$$\vartheta(y_1,y_2,\ldots)=\sum\sum\cdots\prod_m(1-y_m)^{N_m}P(N_1,N_2,\ldots),\quad \forall|1-y_m|\leqslant 1,\quad (A23)$$

$$\tilde{\vartheta}(y_1,y_2,\ldots)=\sum\sum\cdots\prod_m(1-y_m)^{N_m}\tilde{P}(N_1,N_2,\ldots),\quad \forall|1-y_m|\leqslant 1.\quad (A24)$$

By comparing Eqs. (A23) and (A24) with the discrete versions of the definitions of the characteristic functionals $G[K(\mathbf{x})]$ and $\tilde{G}[k(\mathbf{x})]$,

$$G[K(\mathbf{x})]=\sum_{N_1}\sum_{N_2}\cdots P(N_1,N_2,\ldots)\exp\left(\sum_m iK(\mathbf{x}_m)N_m\right),\quad (A25)$$

$$\tilde{G}[K(\mathbf{x})]=\sum_{N_1}\sum_{N_2}\cdots \tilde{P}(N_1,N_2,\ldots)\exp\left(\sum_m iK(\mathbf{x}_m)N_m\right),\quad (A26)$$

we notice that

$$G[K(\mathbf{x})]=\vartheta[y(\mathbf{x})=1-\exp(iK(\mathbf{x}))].\quad (A27)$$

$$\tilde{G}[K(\mathbf{x})]=\tilde{\vartheta}[y(\mathbf{x})=1-\exp(iK(\mathbf{x}))].\quad (A28)$$

Combining Eq. (A22) with Eqs. (A27) and (A28), we obtain the renormalization group equation (26).

APPENDIX B

By expanding the exponential in Eq. (24) in a functional Taylor series, we obtain

$$l(t)=\exp\left(-\int\rho(W)dW\right)\times\left\{1+\sum_{N=1}^{\infty}\frac{1}{N!}\int\cdots\int\rho(W_1)\cdots\rho(W_N)\times\exp\left(-t\sum_{v=1}^{N}W_v\right)dW_1\cdots dW_N\right\}.\quad (B1)$$

Equation (B1) has exactly the same structure as the grand canonical average (53). By comparing Eqs. (B1) and (53), we come to Eqs. (54) and (55). Similarly, by expressing the intermittent limit law (32) in the form

$$l(t)=H\int_0^1 z^{H-1}dz\,\exp\left\{-(1+1/H)z\int\rho(W)\times[1-\exp(-Wt)]dW\right\},\quad (B2)$$

expanding the exponential in Eq. (B2) in a functional Taylor series similar to Eq. (B1) and comparing the result with Eq. (53), we obtain Eqs. (58) and (59).

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